Koszul Duality

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It makes us, or it mars us; think on that,
And fix most firm thy resolution.
— *Othello*

Contents

1 Introduction

This is an essay about some dualities in algebra, and in particular a small cluster of cognate duality phenomena called *Koszul Duality*.

While special cases of this had been known before, it was first codified in a general way in the foundational paper [5] by Priddy. Here, algebras were considered which were given by generators and relations, such that the relations were in some sense quadratic in the generators. These are known as *quadratic algebras*.

The general results yield a recipe for duality, which exchanges algebras with many relations for those with few, and exchanges commutativity conditions for skew-commutativity conditions.

This is useful at several levels. Most mundanely, this might give us a way to deal with things which are more free, or smaller, or more commutative, or more skew-commutative than what we started with.

But this is not all. It also gives many resolutions for such algebras, which allow us to do computations in the field of homological algebra. An example is computing Tor and Ext for modules over them.

Based on Priddy's work, Ginzburg and Kapranov in [1] engineered a substantial generalisation. Their paper involves *linear operads*; these are constructions which describe categories of algebraic objects (here just called *algebras*).

There is a duality for these too: analogously, it exchanges operads describing very tightly curtailed families of algebras with operads describing very arbitrary ones, and exchanges operads describing algebras with commutative multiplication laws with operads describing algebras with skew-commutative multiplication laws.

In section 2, I provide some motivation, and sketch definitions for operads, their algebras and modules. They are sketches in the sense that I have not attempted to distract the reader with the commutative coherence diagrams. I refer the unsatisfied to [4].

Then section 3 introduces the simplest duality phenomenon for operads, *quadratic duality.* We follow [1] quite closely.

Section 4 is more concrete. Working in somewhat less generality, and thus with rather less complications, than the original paper [5], we present a duality theory for graded algebras. As some examples (due to Manin in [3]) suggest, this duality can also be thought of as a duality between geometry and supergeometry, although I am not competent to pursue this link.

We return to operads in section 5. Here we pursue a second duality theory, which appears when all our vector spaces have a differential graded structure. The interplay between this and our earlier theory is a key theme of [1], and some facets of this are discussed in section 6.

The last two sections then deal with practical matters: sections 7 and 8 are concerned with the homological calculations possible from the work of the papers [5] and [1] respectively.

Due to constraints in my space, time and understanding, I have only been able to cover some of the results of the theory. I have tried to ensure:

- that the only real prerequisite is a basic familiarity with abstract algebra and elementary representation theory,
- that the exposition is self-contained if not complete, and
- that proofs are given which cover as many as possible of the main techniques used in the original papers.

I would like to thank Michael Mandell for suggesting the topic, and for his many helpful comments.

In this essay, everything will be done over a base field k of characteristic zero.

2 An introduction to operads

A key notion throughout this essay will be the notion of an *linear operad*. We may think of this as an object that defines a category of algebras with structure.

To start with an analogy, suppose we wish to define some category of vector spaces with special additional structure. We can frequently do so by giving an algebra A, and then demonstrating that the structured vector spaces in question are exactly the modules over A.

Examples are numerous, but here are three simple and standard ones:

- a vector space with a chosen endomorphism is exactly a $\mathbb{C}[T]$-module,
- a vector space with a chosen automorphism is exactly a $\mathbb{C}[T, T^{-1}]$-module,
- a vector space with a chosen involution is exactly a $\mathbb{C}[T]/(T^2 - 1)$-module.

Linear operads are intended to perform the same task for algebras that algebras do for modules: there will be a notion of an *algebra over an operad*, which is an algebra with a certain kind of structure.

Accordingly, we will eventually construct the operads $\mathcal{C}om$, $\mathcal{L}ie$ and $\mathcal{A}s$, whose algebras are exactly the commutative, Lie and associative algebras respectively.

Of course, the motivation is that, just as it is easier to manipulate a ring than its category of modules, it should be easier to manipulate an operad than its category of algebras.

Operads

How is this to be achieved? We will define a k**-linear operad** $\mathcal{C}$ to consist of the following ingredients:

- A k-vector space $\mathcal{C}(n)$ for all $n \geq 1$ (which, when the appropriate definitions have been made, will turn out to parametrise n-ary operations $A^{\otimes n} \to A$, for A a $\mathcal{C}$-algebra),
- A unit $1 \in \mathcal{C}(1)$ (which will represent the identity function $A \to A$),
- An action of the symmetric group S_n on each $\mathcal{C}(n)$ (which will model permutations of the arguments to the n-ary operations making up $\mathcal{C}(n)$), and
- Maps
$$\gamma : \mathcal{C}(i) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_i) \longrightarrow \mathcal{C}(j_1 + \cdots + j_i)$$
for all $i, j_1, \ldots, j_i \geq 0$ (which will turn out to represent composition of functions: we can take a j_m-ary function for each m, and apply an i-ary function to their results, which defines an $(j_1 + \cdots + j_i)$-ary function).

These are subject to a large number of compatibility conditions, which are entirely natural given the suggested interpretations of the data: they consist of unit, associativity and equivariance conditions.

In particular, we want to ensure that the unit really does act as a unit, that any iterated composition is independent of the order in which we compose the parts, and that the action of the symmetric group is respected by permuting the arguments in a composition. They are described in detail, for example, in [4].

Hereafter, if $\mathcal{C}$ is an operad, I will refer to $\mathcal{C}(n)$ as its n**-ary part**.

A morphism of operads $\mathcal{C} \to \mathcal{D}$ is defined in the natural way, as a linear map $\mathcal{C}(n) \to \mathcal{D}(n)$ for each n, compatible with the actions of the symmetric groups, with the composition γ, and preserving the unit.

It is also well worth mentioning that operads (as well as the notions of algebras and modules which follow) can be defined over any abelian category with a reasonable tensor product. While everything that is done early on in the essay will be over the category of vector spaces, later we shall use:

- The category gVect^+ of graded vector spaces over k, with tensor product satisfying the commutativity isomorphism

$$V \otimes W \longrightarrow W \otimes V, \qquad v \otimes w \mapsto w \otimes v.$$

  An operad in this category shall be called a **graded (k-linear) operad**. Note that any k-linear operad $\mathcal{C}$ can be regarded as a graded operad, concentrated in degree 0, which we shall call $\mathcal{C}^+$.

- The category gVect^- of graded vector spaces over k, with tensor product satisfying the commutativity isomorphism

$$V \otimes W \longrightarrow W \otimes V, \qquad v \otimes w \mapsto (-1)^{\deg v \deg w} w \otimes v.$$

  An operad in this category shall be called a **supergraded (k-linear) operad**. Note that any k-linear operad can be regarded as a supergraded operad, concentrated in degree 0, which we shall call $\mathcal{C}^-$.

- The category of differential graded vector spaces over k, where the underlying graded structure is actually what we have called supergraded, and the differential decreases degree by 1. Operads in this category shall be called **DG (k-linear) operads**. Any supergraded operad can be regarded as a DG operad with zero differential.

Algebras

Now, operads are not going to be any use to us unless I make good my promise to define algebras over them. The basic insight we use to do this is the one mentioned parenthetically in the definition of an operad: that we expect $\mathcal{C}(n)$ to parametrise operations $A^{\otimes n} \to A$.

In particular, an **algebra** A over a k-linear operad $\mathcal{C}$, often called simply a $\mathcal{C}$-algebra, consists of a k-vector space A, equipped with maps

$$f_n : \mathcal{C}(n) \otimes A^{\otimes n} \longrightarrow A,$$

for all $n \geq 0$, satisfying natural unit, associativity and equivariance conditions found, again, in [4].

Now these have been defined, it is high time to present some important examples:

- The trivial operad $\mathcal{T}riv$ has $\mathcal{T}riv(1) = k$ and $\mathcal{T}riv(n) = 0$ for $n > 1$. An algebra A over $\mathcal{T}riv$ is exactly a vector space: the one-dimensional space $\mathcal{T}riv(1)$ of unary operations is just the scalar multiplications:

$$f_1 : \mathcal{T}riv(1) \otimes A = k \otimes A \xrightarrow{\sim} A.$$

- The commutative operad $\mathcal{C}om$ has $\mathcal{C}om(n) \cong k$ for all n. This represents the fact that—up to scalar multiples—there should be only one way to multiply n elements together in a commutative algebra.

  To construct $\mathcal{C}om$, with all its structure, we must however be a little more sensitive than this. We regard $\mathcal{C}om(n)$ as being generated by the monomial $m(n) = x_1 \dots x_n$, with trivial S_n-action, and allow the composition γ to take generators to generators, in the sense that

$$\gamma(m(i), m(j_1), \dots, m(j_i)) = m(j_1 + \cdots + j_i).$$

  It is simple now to check that an algebra over $\mathcal{C}om$ is exactly a non-unital commutative algebra. In particular,

$$xy = f_2(m(2), x, y) = f_2(\sigma(m(2)), y, x) = f_2(m(2), y, x) = yx,$$

  where σ generates S_2.

  The category of non-unital commutative algebras is equivalent to the category of augmented unital commutative algebras: given a non-unital commutative algebra A we can formally adjoin a unit to get $k \oplus A$, with augmentation given by the projection onto the first argument. Conversely, given an augmented algebra, the augmentation ideal is a non-unital algebra.

- To construct the Lie operad $\mathcal{L}ie$, we take as $\mathcal{L}ie(n)$ the vector space spanned by all bracketings of the monomials $x_1, \dots, x_n$, in any order, equipped with the natural S_n-action by permuting the variables. Anticommutativity and the Jacobi identity imply that these are not all independent.

  In other words, in line with the preceding example, $\mathcal{L}ie(n)$ is the subspace of the free Lie algebra on n generators, which has degree 1 in each generator.

  For example, $\mathcal{L}ie(1) \cong k$, generated by x_1, and $\mathcal{L}ie(2) \cong k$, generated by $[x_1, x_2]$. By repeated use of anticommutativity, $\mathcal{L}ie(3)$ is spanned by the three bracket monomials

$$[x_1, [x_2, x_3]], \quad [x_2, [x_3, x_1]] \quad \text{and} \quad [x_3, [x_1, x_2]],$$

  but these span only a two-dimensional vector space, being linearly dependent by the Jacobi identity.

The composition map γ is the obvious composition of bracket monomials, and an algebra over $\mathcal{L}ie$ is exactly a Lie algebra.

- To construct the associative operad $\mathcal{A}s$, the same technique works: we take $\mathcal{A}s(n)$ to be the vector space spanned by all $x_{\sigma(1)} \ldots x_{\sigma(n)}$, for $\sigma \in S_n$, with the natural S_n-action.

  Plainly, $\mathcal{A}s(n) \cong kS_n$ as kS_n-modules (and thus $\dim_k \mathcal{A}s(n) = n!$), and the definition of γ is as straightforward as before. As one would expect, an $\mathcal{A}s$-algebra is exactly a non-unital associative algebra.

  As is the case for commutative algebras, the category of algebras that we end up with is equivalent to the category of augmented associative algebras.

Finally, the definition of a morphism of algebras is clear: it's a linear map commuting with all the f_n.

Modules

In addition to algebras, there is a good general notion of a module over an algebra over an operad. For $\mathcal{C}$ an operad, and A a $\mathcal{C}$-algebra, we define a $(\mathcal{C}, A)$**-module** to be a k-vector space M equipped with maps

$$g_n : \mathcal{C} \otimes A^{\otimes(n-1)} \otimes M \longrightarrow M,$$

satisfying the unit, associativity and equivariance conditions I am loath to describe.

It is, however, to be mentioned here that a consequence of the equivariance is that $(\mathcal{C}, A)$-modules are always two-sided. So, while a $(\mathcal{C}om, A)$-module is just an ordinary A-module, an $(\mathcal{A}s, A)$-module is an A-bimodule. A $(\mathcal{L}ie, A)$-module is a Lie algebra representation of A.

Now, equipped with these definitions, and the most famous examples, we are ready to study operads and their algebras in a systematic way.

3 Quadratic operads and quadratic duality

Following all the generality of the last section, I now confess that the notion of operad is too general to have many exciting properties. We will now systematically remove some of this generality, to restrict ourselves to a class of operads which is general enough to contain all the examples mentioned so far, but specific enough to have a beautiful theory.

The observation which makes this possible is that all the operads mentioned so far are, in some sense, generated by their unary and binary parts. Indeed, in manipulating associative, Lie, and commutative algebras, the only operation that one ever needs to write down is the binary product operation.

Furthermore, in all these examples, all the relations that these binary operations satisfy are follow from the relations using only two or three variables.

So, for example, $\mathcal{A}s$ is, in this still-vague sense, generated by the product operation, subject to the relation $(x_1x_2)x_3 = x_1(x_2x_3)$ and those which are in its S_3-orbit. $\mathcal{C}om$ is, again, generated by the product operation, but this is now subject to the relation $x_1x_2 = x_2x_1$. $\mathcal{L}ie$ is generated by the bracket, and subject only to anticommutativity and the Jacobi identity, relations in two and three variables respectively.

We now seek to explain what it means for an operad $\mathcal{C}$ to be generated by $\mathcal{C}(1)$ and $\mathcal{C}(2)$. As usual, this should mean that it is a quotient of a free operad generated by $\mathcal{C}(1)$ and $\mathcal{C}(2)$. But this gives us two concepts that we should define.

Free quadratic operads

Observe, to start with, that if $\mathcal{C}$ is a k-linear operad, then $\mathcal{C}(1)$ is a k-algebra, with product defined by

$$\gamma : \mathcal{C}(1) \otimes \mathcal{C}(1) \longrightarrow \mathcal{C}(1)$$

and unit given by $1 \in \mathcal{C}(1)$. Hereafter, we call this algebra K.

Furthermore, $\mathcal{C}(n)$ is a $(K, K^{\otimes n})$-bimodule (the former acts on the result of the n-ary operations in $\mathcal{C}(n)$, the latter acts on the arguments).

So suppose we have a semisimple k-algebra K and a $(K, K^{\otimes 2})$-bimodule E of finite type with an involution σ.

We proceed to define an operad $\mathcal{Q}(K, E)$, the **free quadratic operad** generated by K and E. This has $\mathcal{Q}(K, E)(1) = K$ and $\mathcal{Q}(K, E)(2) = E$. The higher spaces are defined inductively by the formula

$$\mathcal{Q}(K, E)(n) = \bigoplus_{a+b=n, 1\leq a\leq b} \mathrm{Ind}_{G_{a,b}}^{S_n} \left(E \otimes_{K^{\otimes 2}} \left(\mathcal{Q}(K, E)(a) \otimes_k \mathcal{Q}(K, E)(b) \right) \right),$$

where:

- $G_{a,b}$ is the subgroup of S_n which preserves the partition

$$\{\{1, \ldots, a\}, \{a + 1, \ldots, n\}\}.$$

Explicitly, if $a < b$, this is $S_a \times S_b$, and it acts trivially on E and has a natural action on the right-hand bracket. If $a = b$, then it is the wreath product $S_a \wr C_2$. This, again, may be taken to act in the usual way on E (ie. as the involution) and acts in the natural way on the right-hand bracket (to be particular, the quotient action of C_2 exchanges the factors).

- The $K^{\otimes 2}$-module structure on the right-hand bracket is obtained as the external product of the left K-module structures on each factor.

Perhaps I should motivate this formula a little. We can view n-ary operations in a quadratic operad as linear combinations of rooted directed binary trees with n leaves. Each vertex represents a binary operation from E, and the composition γ of operations is represented by attaching trees together root-to-leaf.

Now, every nontrivial rooted binary tree consists of two rooted binary subtrees, with their roots joined together.

The direct sum in the formula keeps track of the different possible sizes of these two subtrees. The induction provides that the arguments can be split among the leaves of either side in all possible ways. Then the tensor products merely gather all of the possibilities for each of these subtrees, and all the possibilities for combining them.

I shall not go to the trouble of defining γ explicitly for a free quadratic operad, but I will supply some hints. We would base our argument on the fact that such an operad is freely generated by its unary and binary operations. By induction, therefore, we only need to by able to combine two general operations with a binary one, or one with a unary one. Both are possible, and just require appropriate K-module structures which can easily be found.

In [1], a more general construction is given for free operads, that does not require the generators to be quadratic. This will be useless for our purposes, so I have avoided the technicalities they require to do this. In particular, I have not needed to describe the language of trees.

Quotients

Now, we introduce the correct notion of a quotient of an operad. We define an **ideal** $\mathcal{I}$ in an operad $\mathcal{C}$ to be a collection of subspaces $\mathcal{I}(n) \subset \mathcal{C}(n)$, closed under the action of S_n. We also require that, in the map

$$\gamma : \mathcal{C}(i) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_i) \longrightarrow \mathcal{C}(j_1 + \cdots + j_i),$$

if any of the arguments are in the appropriate $\mathcal{I}(n)$, then so is the result.

Given an operad $\mathcal{C}$ and an ideal $\mathcal{I}$ in $\mathcal{C}$, we can form the **quotient operad** $\mathcal{C}/\mathcal{I}$, which has $\mathcal{C}/\mathcal{I}(n) = \mathcal{C}(n)/\mathcal{I}(n)$, with the definitions of 1 and γ inherited from $\mathcal{C}$.

Analogously to standard notation for ideals of rings, when the ambient operad is clear, I shall use the notation $\langle V \rangle$ to mean the smallest ideal containing V. This is, of course, the one generated from V, by applications of the symmetric group actions, and by using γ.

At last we are able to discuss quadratic operads in general. Note that our choice of E, and in particular its action by S_2, will reflect the 2-variable relations satisfied by the operad we construct, so all we need to do is deal with the 3-variable relations.

From the formula in the preceding part, given K and E as we had there, we have

$$\mathcal{Q}(K,E)(3) = \mathrm{Ind}_{S_2}^{S_3}(E \otimes_K E)$$

where the left factor of $K^{\otimes 2}$ acts standardly, and the right factor acts trivially.

Now suppose R is a sub-$(K, K^{\otimes n})$-bimodule of $\mathcal{Q}(K,E)(3)$, which is closed under the action by S_3. We define the **quadratic operad** $\mathcal{Q}(K,E,R)$ to be:

$$\mathcal{Q}(K,E,R) = \mathcal{Q}(K,E)/\langle R \rangle .$$

By remarks made already, the operads $\mathcal{C}om$, $\mathcal{L}ie$ and $\mathcal{A}s$ are quadratic.

Quadratic duality

So far, all we have done is defined certain concepts, commented on their coherence, and noted that the objects intended to satisfy the definitions actually do. Here is the first section of the essay where an elegant result is proved. But there is one last definition to get through first.

What I shall do now is introduce a natural duality theory on quadratic operads. Suppose $\mathcal{C} = \mathcal{Q}(K,E,R)$ is a quadratic operad.

Firstly, observe that $E^\vee = \hom(E,K)$ is a $(K^{\mathrm{op}}, K^{\mathrm{op}\otimes 2})$-bimodule, with structure defined by

$$(x_0 f(x_1 \otimes x_2))(z) = f(x_0 z (x_1 \otimes x_2)),$$

where $x_0, x_1, x_2 \in K$, $z \in E$ and $f \in E^\vee$.

If V is a left K-module with an action of S_n, we give $V^\vee$ the transposed action, tensored with the sign representation.

Secondly, observe that there is a canonical duality pairing between

$$\mathcal{Q}(K,E)(3) = \mathrm{Ind}_{S_2}^{S_3}(E \otimes_K E)$$

and

$$\mathcal{Q}(K^{\mathrm{op}},E^\vee)(3) = \mathrm{Ind}_{S_2}^{S_3}\left(E^\vee \otimes_{K^{\mathrm{op}}} E^\vee\right) = \mathcal{Q}(K,E)(3)^\vee.$$

This means that we have orthogonal complements between the two; hence we can define the **quadratic dual** of $\mathcal{C}$ as:

$$\mathcal{C}^! = \mathcal{Q}(K^{\mathrm{op}},E^\vee,R^\perp),$$

where $R^\perp$ is the space of functions in $\mathcal{Q}(K,E)(3)^\vee \cong \mathcal{Q}(K^{\mathrm{op}},E^\vee)(3)$ which vanish on R.

Plainly (at least when all the vector spaces are finite) we have $\mathcal{C}^{!^!} \cong \mathcal{C}$.

Now we can state and prove that long-awaited theorem:

Theorem 1 *$\mathcal{C}om^! \cong \mathcal{L}ie$, and $\mathcal{A}s^! \cong \mathcal{A}s$.*

Proof: A simplifying feature of these calculations is that all three operads have unary part k, so we can ignore the module structures on the higher parts.

Furthermore, we can do the duality of $\mathcal{C}om$ and $\mathcal{L}ie$ just with representation theory: since none of the representations we will consider have more than one copy of any irreducible representation, describing a subrepresentation up to isomorphism specifies it uniquely.

Let the trivial representation of a symmetric group be 1, the sign representation S, the "reduced" standard representation V (so that the standard representation is $1 \oplus V$), and the regular representation C.

It is easy to check that, as representations, we have:

$$\mathcal{C}om(1) = 1, \qquad \mathcal{C}om(2) = 1, \qquad \mathcal{C}om(3) = 1,$$
$$\mathcal{L}ie(1) = 1, \qquad \mathcal{L}ie(2) = S, \qquad \mathcal{L}ie(3) = V.$$

Furthermore, $\mathcal{Q}(k,1)(3) = 1 \oplus V$ is the usual permutation representation of S_3. Concretely, S_3 acts by permuting the variables in

$$\{x_1(x_2x_3), x_2(x_3x_1), x_3(x_1x_2)\},$$

assuming commutativity. Thus $\mathcal{C}om = \mathcal{Q}(k, 1, V)$.

Similarly, $\mathcal{Q}(k, V)(3) = S \oplus V$, with S_3 acting by permuting the variables in

$$\{[x_1, [x_2, x_3]], [x_2, [x_3, x_1]], [x_3, [x_1, x_2]]\},$$

assuming anticommutativity. Thus $\mathcal{L}ie = \mathcal{Q}(k, S, S)$.

Remembering to tensor by the sign representation, these match up properly (and uniquely) and the first part follows.

Now, as representations,

$$\mathcal{A}s(1) = 1, \qquad \mathcal{A}s(2) = 1 \oplus S, \qquad \mathcal{A}s(3) = C = 1 \oplus S \oplus V^{\oplus 2}.$$

The representation $\mathcal{Q}(k, 1 \oplus S)(3) = C^{\oplus 2}$, with S_3 acting by permutation on the twelve monomials of the forms $x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)})$ and $(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}$.

Following [1], we introduce a quadratic form q on this, by taking these monomials to be orthogonal and letting the former ones have norm $\operatorname{sgn}(\sigma)$ and the latter $-\operatorname{sgn}(\sigma)$, and checking that the space of relations, spanned by the six associators, is its own annihilator. This completes the proof. $\square$

Products for quadratic operads

Having seen that there is some fun to be had by manipulating quadratic operads, now we will deal with some further operations on them.

A natural question, whenever we have defined a kind of algebraic object, is "in what ways can one define products of such objects?" In this section we exhibit several good answers in the category of quadratic operads, and demonstrate their relevance.

Firstly, and most simply, given two operads $\mathcal{C}_1, \mathcal{C}_2$ of any sort, we can form their **tensor product**, $\mathcal{C}_1 \otimes \mathcal{C}_2$, with $(\mathcal{C}_1 \otimes \mathcal{C}_2)(n) = \mathcal{C}_1(n) \otimes \mathcal{C}_2(n)$. The correct definitions of composition, units and actions by the symmetric group are all obvious.

A pleasant property of the tensor product is that if A_1 is a $\mathcal{C}_1$-algebra and A_2 is a $\mathcal{C}_2$-algebra, then $A_1 \otimes A_2$ naturally has a structure as an $\mathcal{C}_1 \otimes \mathcal{C}_2$-algebra.

It is immediate that

$$\mathcal{Q}(K_1, E_1, R_1) \otimes \mathcal{Q}(K_2, E_2, R_2) = \mathcal{Q}(K_1 \otimes K_2, E_1 \otimes E_2, R_1 \otimes R_2).$$

We can immediately define another product operation for quadratic operads:

$$\mathcal{C}_1 \hat{\otimes} \mathcal{C}_2 = (\mathcal{C}_1^! \otimes \mathcal{C}_2^!)^!,$$

which is such that

$$\begin{aligned} & \mathcal{Q}(K_1, E_1, R_1) \hat{\otimes} \mathcal{Q}(K_2, E_2, R_2) \\ = \; & \mathcal{Q}(K_1 \otimes K_2, E_1 \otimes E_2, (R_1^\perp \otimes R_2^\perp)^\perp) \\ = \; & \mathcal{Q}(K_1 \otimes K_2, E_1 \otimes E_2, (R_1 \otimes F_2) + (F_1 \otimes R_2)), \end{aligned}$$

where $F_i = \mathcal{Q}(K_i, E_i)(3)$ is the total 3-ary part of the free quadratic operad corresponding to $\mathcal{C}_i$.

Restricting to quadratic operads whose unary part is k, there are two further products. By considering the formula given above for the 3-ary parts F_i of the free quadratic operad $\mathcal{Q}(K_i, E_i)$, we observe that they contain three copies of $E_i \otimes E_i$. Allow P_i to denote one of these copies. We can then define the **circle products**:

$$\begin{aligned} \mathcal{C}_1 \circ \mathcal{C}_2 &= \mathcal{Q}(k, E_1 \otimes E_2, (P_1 \otimes R_2) + (R_1 \otimes P_2)) \\ \mathcal{C}_1 \bullet \mathcal{C}_2 &= \mathcal{Q}(k, E_1 \otimes E_2, (P_1 \otimes R_2) \cap (R_1 \otimes P_2)) \end{aligned}$$

These products have some very interesting properties, and mesh well with quadratic duality, as well as with $\mathcal{C}om$ and $\mathcal{L}ie$, thus:

Theorem 2 *The two circle products are related by*

$$\mathcal{C}_1^! \circ \mathcal{C}_2^! = (\mathcal{C}_1 \bullet \mathcal{C}_2)^!.$$

Furthermore, $\mathcal{C}om$ is a unit for $\circ$ and $\mathcal{L}ie$ is a unit for $\bullet$.

Proof: A verification of the first identity runs as follows:

$$\begin{aligned} (\mathcal{C}_1^! \circ \mathcal{C}_2^!)^! &= (\mathcal{Q}(k, E_1^\vee, R_1^\perp) \circ \mathcal{Q}(k, E_2^\vee, R_2^\perp))^! \\ &= (\mathcal{Q}(k, E_1^\vee \otimes E_2^\vee, (P_1^\vee \otimes R_2^\perp) + (R_1^\perp \otimes P_2^\vee)))^! \\ &= \mathcal{Q}(k, E_1 \otimes E_2, ((P_1^\vee \otimes R_2^\perp) + (R_1^\perp \otimes P_2^\vee))^\perp) \\ &= \mathcal{Q}(k, E_1 \otimes E_2, ((P_1^\vee \otimes R_2^\perp)^\perp \cap (R_1^\perp \otimes P_2^\vee)^\perp)) \\ &= \mathcal{Q}(k, E_1 \otimes E_2, (P_1 \otimes R_2) \cap (R_1 \otimes P_2)) \\ &= \mathcal{C}_1 \bullet \mathcal{C}_2. \end{aligned}$$

The following calculation, in which we freely use results from the proof of Theorem 1, shows that $\mathcal{C}om$ is a unit for $\circ$:

$$\begin{aligned} \mathcal{C}om \circ \mathcal{Q}(k, E, R) &= \mathcal{Q}(k, 1 \otimes E, (1 \otimes R) + (V \otimes (E \otimes E))) \\ &= \mathcal{Q}(k, 1 \otimes E, (1 \otimes R)) = \mathcal{Q}(k, E, R), \end{aligned}$$

since $V \otimes (E \otimes E)$ is trivial, as S_3 acts symmetrically on $E \otimes E$.

Now the result for $\mathcal{L}ie$ follows purely formally from these two:

$$\mathcal{L}ie \bullet \mathcal{C} = (\mathcal{L}ie^! \circ \mathcal{C}^!)^! = (\mathcal{C}om \circ \mathcal{C}^!)^! = \mathcal{C}^{!^!} = \mathcal{C}.$$

This completes the proof. □

4 Quadratic algebras and quadratic duality

Quadratic algebras

It is time to consider applications of the theory we have discussed. Thus, we construct an analogue of quadratic operads in the category of algebras over a quadratic operad: the notion of *quadratic algebra*.

Just as a quadratic operad is generated by its binary part, subject only to 3-ary relations, a quadratic algebra will be canonically internally graded, generated by its elements of degree 1, and subject to relations which are homogenous in degree 2.

As such the construction will be entirely analogous: we will first construct a *free quadratic algebra*, then discuss ideals and quotients in algebras, and then make the obvious definition.

However, there is a complication that does not arise in the case of operads: if $\mathcal{C}$ is a k-linear operad, then, as has been observed in section 2 above, we can regard it either as a graded operad or as a supergraded operad, concentrated in degree zero. Recall that these will be denoted by $\mathcal{C}^+$ and $\mathcal{C}^-$ respectively. Also, from here on, I will write $\mathcal{D}$-Alg for the category of algebras over $\mathcal{D}$.

So, in this section, $\mathcal{C} = \mathcal{Q}(K, E, R)$ is a quadratic operad, and we recall that the smallprint of my definition stated that K was to be a semisimple k-algebra.

If V is a K-module, which we will regard as graded/supergraded, and concentrated in degree 1, we define the **free quadratic algebras** $A^+_{\mathcal{C}}(V)$ and $A^-_{\mathcal{C}}(V)$ to consist of graded parts $A^+_{\mathcal{C},n}(V)$ and $A^-_{\mathcal{C},n}(V)$ with

$$A^+_{\mathcal{C},n}(V) = \left(\mathcal{C}^+(n) \otimes_{K^{\otimes n}} V^{\otimes n}\right)_{S_n}, \qquad A^-_{\mathcal{C},n}(V) = \left(\mathcal{C}^-(n) \otimes_{K^{\otimes n}} V^{\otimes n}\right)_{S_n}$$

where $W_{S_n} = W/\langle(\sigma - id)W : \sigma \in S_n\rangle$ is the module of coinvariants of W under the S_n action. The algebra structure is simple: the structure map

$$\mathcal{C}(n) \otimes A^{\pm}_{\mathcal{C}}(V)^{\otimes n} \longrightarrow A^{\pm}_{\mathcal{C}}(V)$$

is defined on homogeneous parts in the natural way as

$$\begin{array}{cl} & \mathcal{C}(n) \otimes A^{\pm}_{\mathcal{C},i_1}(V) \otimes \cdots \otimes A^{\pm}_{\mathcal{C},i_n}(V) \\ = & \mathcal{C}(n) \otimes \left(\mathcal{C}(i_1) \otimes_{K^{\otimes i_1}} V^{\otimes i_1}\right)_{S_{i_1}} \otimes \cdots \otimes \left(\mathcal{C}(i_n) \otimes_{K^{\otimes i_n}} V^{\otimes i_n}\right)_{S_{i_n}} \\ \longrightarrow & \left(\mathcal{C}(n) \otimes_{K^{\otimes n}} V^{\otimes n}\right)_{S_n} \\ = & A^{\pm}_{\mathcal{C},i_1+\cdots+i_n}(V). \end{array}$$

Thus we are justified in saying that $A^{\pm}_{\mathcal{C}}(V)$ is a $\mathcal{C}^{\pm}$-algebra.

An **ideal** of a $\mathcal{C}$-algebra A is a subspace $I \subset A$ closed under all operations, in the sense that:

$$f_n(\mathcal{C}(n) \otimes A^{\otimes(n-1)} \otimes I) \subset I.$$

This clearly agrees with the usual notion of a (two-sided) ideal in the operads $\mathcal{C}om$, $\mathcal{L}ie$ and $\mathcal{A}s$.

As for operads, I will let $\langle S \rangle$ denote the smallest ideal containing S, whenever the ambient algebra is understood.

If we have a $\mathcal{C}$-algebra A, and an ideal I, then we can define the **quotient algebra** A/I in the obvious way.

Thus, if we have a K-module V, and a K-submodule of relations

$$S \subset A^{\pm}_{\mathcal{C},2} = (\mathcal{C}(2) \otimes_{K^{\otimes 2}} V^{\otimes 2})_{S_2} = (E \otimes_{K^{\otimes 2}} V^{\otimes 2})_{S_2},$$

where the S_2 action is given by

$$\sigma(e \otimes (v_1 \otimes v_2)) = \pm\sigma(e \otimes (v_2 \otimes v_1)),$$

then we define the **quadratic algebra** $A_{\mathcal{C}}(V, S)$ to be $A_{\mathcal{C}}(V)/\langle S \rangle$.

Quadratic duality for algebras

We can now define a quadratic duality for algebras, analogous to that for operads introduced earlier.

Now, quadratic duality for algebras is a correspondence

$$\mathcal{C}^{+}\text{-Alg} \longleftrightarrow (\mathcal{C}^{!})^{-}\text{-Alg}.$$

In other words, it turns graded associative algebras into supergraded associative algebras (which was, in fact, the original quadratic duality), supergraded commutative algebras into graded Lie algebras, and so on.

It is defined (in one direction) by $A^+_{\mathcal{C}}(V,S) = A^-_{\mathcal{C}^!}(V^\vee, S^\perp)$, where $V^\vee = \hom(V,K)$ is, as required, a K^{op}-module, and

$$\begin{aligned} S^\perp &\subset A^+_{\mathcal{C},2}(V)^\vee \\ &= (E \otimes_{K^{\otimes 2}} V^{\otimes 2})^\vee_{S_2} \\ &= (E^\vee \otimes_{K^{\mathrm{op}\otimes 2}} V^{\vee\otimes 2})_{S_2} \\ &= A^-_{\mathcal{C},2}(V^\vee). \end{aligned}$$

Note in particular that, as operads, $\mathcal{A}s^+ \cong \mathcal{A}s^-$. So graded associative algebras are the same thing as supergraded ones, and so quadratic duality on $\mathcal{A}s$ is a duality of associative algebras with associative algebras. This duality theory is the one developed in [5].

Here are some examples:

- Viewed as a graded associative algebra, the polynomial algebra on n generators $k[x_1,\ldots,x_n]$ has as its quadratic dual the exterior algebra on n generators $k[\theta_1,\ldots,\theta_n]$.

  To see this, observe that the former algebra is generated by $V = \langle x_1,\ldots,x_n\rangle$ with space of relations $S = \langle x_ix_j - x_jx_j\rangle_{1\le i,j\le n}$, which is a subspace of dimension $n(n-1)/2$ of $V^{\otimes 2} \cong (\mathcal{A}s(2)\otimes V^{\otimes 2})_{S_2}$.

  A generating set for its quadratic dual can thus be written $V^\vee = \langle\theta_1,\ldots,\theta_n\rangle$. The natural pairing is $\langle x_i,\theta_j\rangle = \delta_{ij}$. This extends to the pairing $V^{\otimes 2}\otimes V^{\vee\otimes 2} \to k$, as

  $$\left\langle x_i\otimes x_{i'}, \theta_j\otimes\theta_{j'}\right\rangle = \delta_{ij}\delta_{i'j'}.$$

  We can now calculate its quadratic dual directly: I shall exhibit some elements in the annihilator of S:

  $$S^\perp = \langle\theta_i\theta_j + \theta_j\theta_i\rangle_{1\le i,j\le n} + \langle\theta_i\theta_i\rangle_{1\le i\le n}$$

  which has dimension $n(n+1)/2$, as required, completing the argument.

- To take another example over the operad $\mathcal{A}s$, Manin, in [3], makes use of the algebras,

  $$\begin{aligned} \mathbb{A}^{2|0}_q &= k\langle x,y\rangle / \langle xy = qyx\rangle \quad \text{and} \\ \mathbb{A}^{0|2}_q &= k\langle\eta,\xi\rangle / \left\langle \eta^2 = \xi^2 = \eta\xi + q\xi\eta = 0\right\rangle, \end{aligned}$$

  where q is a fixed nonzero element of k. He sees these “quantum plane” algebras as forming part of a much more general geometric interpretation of the quadratic duality. But this would take me both out of my depth and out of the central theme of this essay.

However, interpreting the former as an $\mathcal{A}s^+$-algebra and the latter as an $\mathcal{A}s^-$-algebra, what we certainly are qualified to say is that these are quadratic duals of one another. Indeed, the relations specified can be made to be orthogonal under a sensible choice of pairing. We let dual bases for V and $V^\vee$ be $\{x, y\}$ and $\{\eta, \xi\}$ and thus dual bases for $A_{\mathcal{C},2}(V)$ and $A_{\mathcal{C},2}(V^\vee)$ are $\{x^2, y^2, xy, yx\}$ and $\{\eta^2, \xi^2, \xi\eta, \eta\xi\}$. Then:

$$\begin{aligned}\langle xy - qyx, \eta\xi + q\xi\eta\rangle &= \left\langle xy - qyx, \eta^2\right\rangle \\ &= \left\langle xy - qyx, \xi^2\right\rangle = 0,\end{aligned}$$

which proves the duality.

5 DG operads and DG cobar duality

In this section we introduce a second duality for operads. This time we (temporarily) abandon our earlier restriction to quadratic operads; the sophistication this time lies in using differential graded operads.

I will, as usual, write gradings as subscripts if they decrease degree by 1, and as superscripts if they increase it by 1.

We will impose a technical condition: an operad $\mathcal{C}$ is **admissible** if each $\mathcal{C}(n)$ is a finite vector space, and $\mathcal{C}(1)$ is a semisimple k-algebra concentrated in degree 0.

A key part of the construction involves setting up a collection of complexes, the *cobar complexes*, whose first terms are the n-ary parts of the operad. We regard these complexes as bigraded modules. For clarity, we will call the former grading, which arises from the DG structure of the operads being considered, the *DG grading* and the latter one, which we impose explicitly, the *composition grading*.

This is analogous to the cobar construction in classical homological algebra. There, we (roughly speaking) associate to an algebra A a graded object whose n-th degree part is the space of formal products of n elements from A.

Now, the n-th degree part of the cobar complex of an operad $\mathcal{C}$ shall be the space of formal combinations of n elements of the operad. To make sense of this, we will need some combinatorial machinery.

Trees and free operads

Define $T(m,n)$ to be the set of based trees, directed away from the base with m unlabelled interior vertices and n labelled leaves, where every interior vertex has outgoing degree at least two.

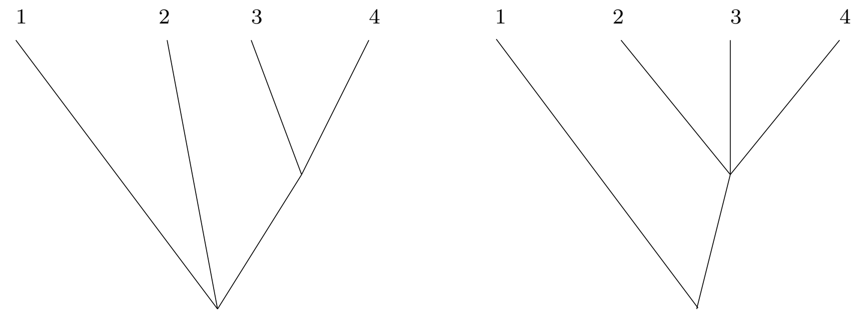

Figure 1: Example: all elements of $T(2,4)$ can be obtained from these two by relabelling the leaves.

Now given an operad $\mathcal{C}$ and a tree $t \in T(m,n)$, we define

$$t(\mathcal{C}) = \bigotimes_{v\in t} \mathcal{C}(\mathrm{OutDeg}(v)).$$

This should be thought of as the space of ways of labelling each vertex of t with an appropriate element of $\mathcal{C}$.

As a useful first example of the relevance of trees to the theory of operads, we shall use them to construct a notion of free operad more general than that used in section 3 above.

Let K be a semisimple k-algebra, and let $E(n)$, for $n \geq 2$, be a family of $(K, K^{\otimes n})$-bimodules, each with an action of S_n.

Define the **free operad** $\mathcal{F}(E)$ to have n-ary part

$$\mathcal{F}(E)(n) = \bigoplus_{0<m<n, t\in T(m,n)} \left(\bigotimes_{v\in t} E(\mathrm{OutDeg}(v)) \right),$$

for $n \geq 2$, setting $\mathcal{F}(E)(1)$ to be K. The tensor product is taken over K, in the natural way according to the tree structure. In other words, where two vertices are linked by an edge, K acts on the right of the $E(n)$ corresponding to the source vertex (as the appropriate part of the right $K^{\otimes n}$-module structure), and on the left of the $E(n)$ corresponding to the target vertex (due to its left K-module structure).

The composition operation and maps by S_n are naturally defined, by connecting trees leaf-to-root, and by permuting the leaves respectively. An example is depicted in figure 2.

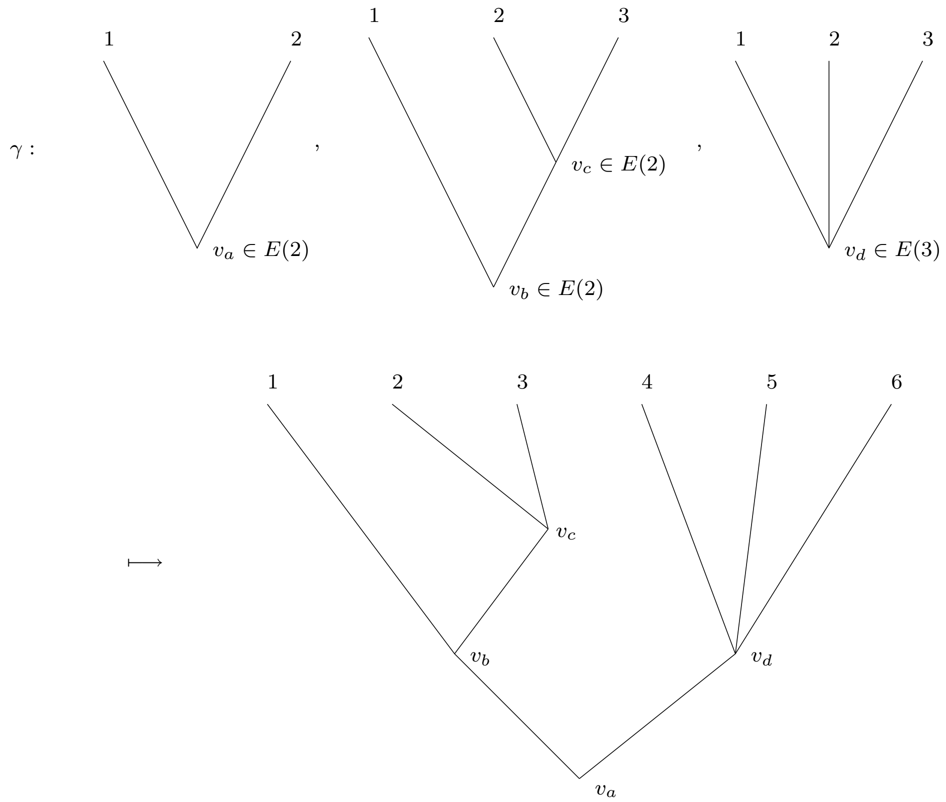

Figure 2: An example of composition of some basis elements of a free operad (here, it is $\gamma : \mathcal{F}(E)(2) \otimes \mathcal{F}(E)(3) \otimes \mathcal{F}(E)(3) \to \mathcal{F}(E)(6)$).

The bar construction and graded dual

Now, using the notation of trees above, we can define the n-th **bar complex**:

$$\bar{\mathcal{C}}(n)_{*,s} = \bigoplus_{t \in T(s,n)} \left(t(\mathcal{C}) \otimes \det(E(t))\right),$$

where $E(t)$ is the vector space freely generated by the edges of t (placed in degree 0), and $\det E(t)$ is the determinant vector space of $E(t)$.

Notice that, since $T(1,n)$ is a singleton, $\bar{\mathcal{C}}(n)_{*,s} = \mathcal{C}(n)$. Also notice that, by the degree requirements, $T(m,n) = 0$ if $m \geq n$, so the complex has composition degree.

Of course, I say "complex", but I have yet to put a differential on it. There is already a differential d of bidegree $(-1, 0)$ due to the DG structure, and so the aim is to create a double complex with another differential δ of bidegree $(0, -1)$.

We can define $\delta : \bar{\mathcal{C}}(n)_{r,s+1} \to \bar{\mathcal{C}}(n)_{r,s}$ by giving its matrix elements; that is to say, a system of linear maps between the direct summands

$$\delta_{t',t} : t'(\mathcal{C}) \otimes \det(E(t')) \to t(\mathcal{C}) \otimes \det(E(t)),$$

for all $t' \in T(s+1, n), t \in T(s, n)$.

Define $\delta_{t',t}$ to be zero unless t is obtained from t' by contracting an edge e. In that case map $t'(\mathcal{C})$ to $t(\mathcal{C})$ by the identity outside the vertices of e, and via

$$\gamma \circ (\mathrm{id} \otimes 1 \otimes \cdots \otimes 1 \otimes \mathrm{id} \otimes 1 \otimes \cdots \otimes 1) : \mathcal{C}(i) \otimes \mathcal{C}(j) \to \mathcal{C}(i + j - 1)$$

on those two vertices. Map $\det(E(t'))$ to $\det(E(t))$ by taking

$$e \wedge f_1 \wedge f_2 \wedge \cdots \longmapsto f_1 \wedge f_2 \wedge \cdots .$$

Now we can state clearly what I have been hinting at:

Theorem 3 *The above differentials make $\bar{\mathcal{C}}(n)_{*,*}$ into a double complex. In other words, $d^2 = \delta^2 = 0$ and $d\delta = \delta d$.*

Proof: That $d^2 = 0$ follows formally from the DG structure on $\mathcal{C}$.

To prove $\delta^2 = 0$, we use that

$$\begin{aligned} \delta^2(t) &= \sum_{t'',t'} \delta_{t'',t'} \delta_{t',t} \\ &= \sum_{e_1 \neq e_2 \in E(t)} \delta_{t/e_1/e_2, t/e_1} \delta_{t/e_1, t}, \end{aligned}$$

(where t/e is the tree t after contracting the edge $e \in E(t)$). But the summands are negated if e_1 and e_2 are swapped over. Indeed, the former takes $e_1 \wedge e_2 \wedge f_1 \wedge \cdots$ to $f_1 \wedge \cdots$, but the latter takes $e_1 \wedge e_2 \wedge f_1 \wedge \cdots = -e_2 \wedge e_1 \wedge f_1 \wedge \cdots$ to $-f_1 \wedge \cdots$.

To prove that $d\delta = \delta d$ is elementary graded algebra. $\square$

It is easy to see that $\bar{\mathcal{C}}(n)_{*,s} = 0$ for all $2 \leq n \leq s$, and thus the total complex will be graded finite if all the spaces $\mathcal{C}(n)$ are.

It is also immediate that $\bar{\mathcal{C}}(n)_{r,s}$ has a natural action of S_n, by relabelling the leaves of the trees.

Now, using these facts, we shall take this unwieldy object and produce a new operad from it. Writing

$$\bar{\mathcal{C}}^{\vee}(n)^i \quad \text{for} \quad \bigoplus_{r+s=i} \left(\bar{\mathcal{C}}(n)_{r,s}\right)^{\vee},$$

we define the **graded dual** of $\mathcal{C}$ to be

$$\mathbf{D}(\mathcal{C}) = \bar{\mathcal{C}}^{\vee} \otimes \Lambda.$$

Here Λ is the operad whose n-ary part $\Lambda(n)$ is k placed in degree $(1-n)$, with the sign representation of S_n.

This is an operad: the action by S_n is inherited from that on $\bar{\mathcal{C}}(n)_{*,*}$, and the composition maps are given by connecting trees root-to-leaf (just as in the construction of the general free operad above).

We must now state a theorem to justify my use of the word "dual":

Theorem 4 *The double graded dual* $\mathbf{D}(\mathbf{D}(\mathcal{C}))$ *of an admissible operad* $\mathcal{C}$ *is quasi-isomorphic to* $\mathcal{C}$.

Proof: First off, it is to be observed that this really does make sense: following through the above construction exactly gives a homologically-graded DG operad from a cohomologically-graded DG operad, and vice versa.

Proving this theorem depends on an important observation: that, for $\mathcal{C}$ an admissible operad, $\mathbf{D}(\mathcal{C})$ is a free operad $\mathcal{F}(E)$ generated by the family of S_n modules given by $E(n) = (\mathcal{C}(n) \otimes S)^{\vee}$, where S stands as ever for the sign representation.

By the definition of the free operad, we can think of this as saying that a generating set for $\mathbf{D}(\mathcal{C})(n)$ is given by trees with n leaves, where every vertex of outgoing degree m is labelled by an element of $\mathcal{C}(m)^{\vee} \otimes S$.

Since $((\mathcal{C}(n) \otimes S)^{\vee} \otimes S)^{\vee} \cong \mathcal{C}(n)$, this says that elements of the double graded dual $\mathbf{D}(\mathbf{D}(\mathcal{C}))(n)$ are generated by trees with n leaves, where each vertex of outgoing degree m is labelled by a tree with m leaves, and each vertex of outgoing degree l of that that is labelled with an element of $\mathcal{C}(l)$ (see figure 3).

We can now exhibit $\mathcal{C}$ naturally as a quotient of $\mathbf{D}(\mathbf{D}(\mathcal{C}))$. The quotient map f is defined by using the composition operator of $\mathcal{C}$ to collapse the tree into a single element of the operad.

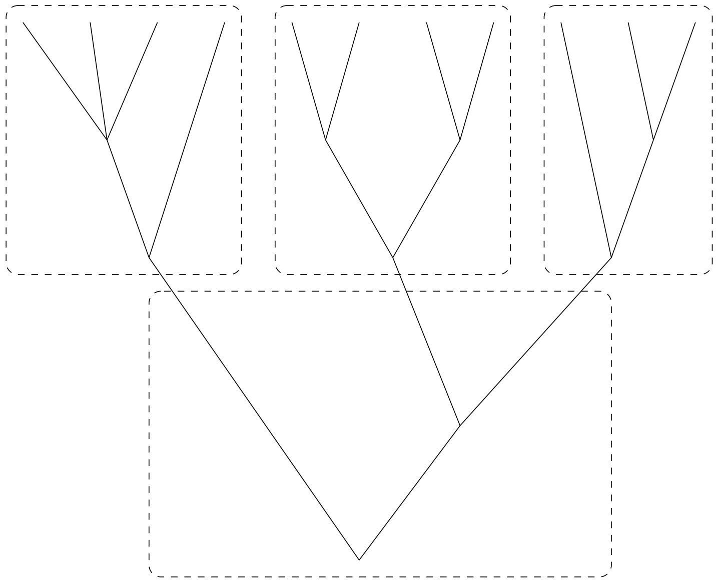

Figure 3: Labelling all the vertices of this object with an appropriate element of $\mathcal{C}$ would yield an element of the double graded dual $\mathbf{D}(\mathbf{D}(\mathcal{C}))$.

Our task is to show f is a quasi-isomorphism, or equivalently that $\ker(f)$ is acyclic. Of course, to do this we must understand the differential d. Since

$$\mathbf{D}(\mathbf{D}(\mathcal{C})) = \left(\operatorname{Tot} \overline{\mathbf{D}(\mathcal{C})_{*,*}}\right)^{\vee} \otimes \Lambda,$$

(where Tot denotes the operation of taking the total complex associated to a double complex), we have $d = d_1 + d'$, where d_1 comes from the *outer* tree structure, and d' is the differential on $\mathbf{D}(\mathcal{C})$.

However, d' in turn comes from a double complex, so $d = d_1 + d_2 + d_3$, where d_2 comes from the *inner* tree structure, and d_3 comes from the DG structure of $\mathcal{C}$ itself.

It suffices to show that $(\mathbf{D}(\mathbf{D}(\mathcal{C}))(n), d_1)$ is acyclic.

The key observation is that giving a tree of trees is equivalent to giving a tree T and a set S of the internal edges in it; that set of edges describes an equivalence relation on the vertices of T, and the equivalence classes are the inner trees (see figure 4).

Since d_1 preserves T, we can consider $\ker(f)$ as the direct sum of a set of complexes $\ker_T(f)$, one for each tree T, and now we wish to show each one of them acyclic.

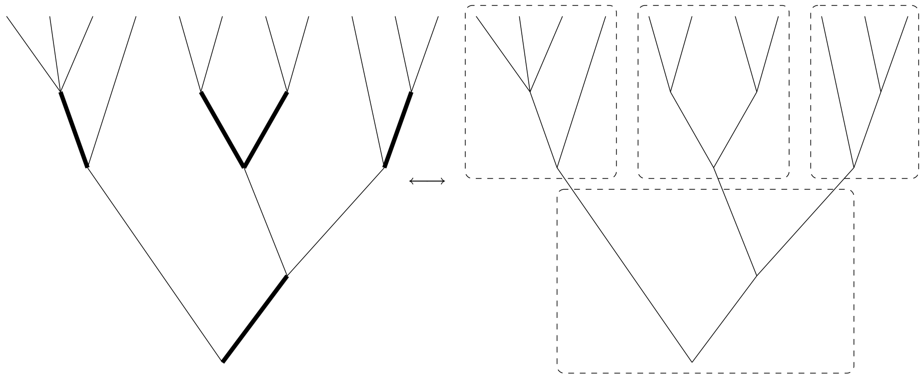

Figure 4: The correspondence: on the left, a tree with a subset of its internal edges (drawn as heavy lines); on the right, the corresponding tree of trees.

But under our correspondence, contracting trees corresponds to removing edges from S, and so we can write:

$$\ker_T(f) = \bigotimes_{v \in T} (0 \to k \to k \to 0)$$

which is clearly acyclic. $\square$

That completes our analysis of the graded dual: we shall go on to show its relevance in the next sections.

6 Koszul operads

This section aims to describe the common ground between the two duality theories for operads that have been introduced.

Thus the framework we work in must permit taking both the quadratic dual and the graded dual; so here we take $\mathcal{C} = \mathcal{Q}(k, E, R)$ a quadratic operad, which we regard as being a DG operad concentrated in degree zero. All such operads are admissible.

By construction, $\mathbf{D}(\mathcal{C})(n)^0$ is the space of binary trees with each internal vertex labelled with an element of $\mathcal{C}(2)^\vee = E^\vee$; in other words, it is isomorphic to the n-ary part of the free operad generated by $E^\vee$:

$$\mathbf{D}(\mathcal{C})(n)^0 \cong \mathcal{Q}(k, E^\vee)(n).$$

This allows us to define a morphism of operads from $\mathbf{D}(\mathcal{C})$ to $\mathcal{C}^!$ as the composition:

$$\begin{array}{rcl}\mathbf{D}(\mathcal{C})(n) & \longrightarrow & \mathbf{D}(\mathcal{C})(n)^0 = \mathcal{Q}(k, E^\vee)(n) = \mathcal{Q}(k^{\mathrm{op}}, E^\vee)(n) \\ & \longrightarrow & \mathcal{Q}(k^{\mathrm{op}}, E^\vee, R^\perp) = \mathcal{C}^!.\end{array}$$

Here, the first map is the identity on the degree-zero part, and the second map is the natural quotient map of operads.

Now, we introduce a key definition: we say that a quadratic operad $\mathcal{C}$ is **Koszul** if the map $\mathbf{D}(\mathcal{C}) \to \mathcal{C}^!$ is a quasi-isomorphism on every part n.

Since $\mathcal{C}^!(n)$ is concentrated in degree zero, this says exactly that $\mathbf{D}(\mathcal{C})(n)$ is a resolution of $\mathcal{C}^!(n)$. The following result shows that being Koszul is a reasonable thing to expect:

Theorem 5

$$H^0(\mathbf{D}(\mathcal{C})(n)) \cong H^0(\mathcal{C}^!(n)) \cong \mathcal{C}^!(n).$$

Proof: The right-hand isomorphism is immediate from the fact that $\mathcal{C}^!(n)$ is concentrated in degree zero.

The left hand isomorphism requires checking. We have

$$\begin{array}{rcl}\mathbf{D}(\mathcal{C})(n)^{-1} & = & \left(\bar{\mathcal{C}}^\vee(n) \otimes \Lambda(n)\right)^{-1} \\ & = & \left(\mathcal{C}^\vee\right)^{n-2} \otimes S.\end{array}$$

This is generated by the trees with one vertex of outgoing degree 3, and all other vertices of outgoing degree 2. Similarly, $\mathbf{D}(\mathcal{C})(n)^0$ is generated by the binary trees.

Since $\mathcal{C}(3)^\vee$ is the space of relations in $\mathcal{C}^!$, the differential maps $\mathbf{D}(\mathcal{C})(n)$ to the space of consequences of relations in $\mathcal{C}^!$. Thus the homology H^0 is $\mathcal{C}^!(n)$: the generators, modulo the consequences of the relations. $\square$

One can regard this theorem as saying that to check that a quadratic operad $\mathcal{C}$ is Koszul we need only check that $\mathbf{D}(\mathcal{C})(n)$ is exact away from degree zero.

Here is a labour-saving result:

Theorem 6 *If $\mathcal{C}$ is Koszul, so is $\mathcal{C}^!$.*

Proof: We use the map $\mathbf{D}(\mathcal{C}) \to \mathcal{C}^!$ constructed at the beginning of the section. Now look at the composite,

$$\mathbf{D}(\mathcal{C}^!) \longrightarrow \mathbf{D}(\mathbf{D}(\mathcal{C})) \longrightarrow \mathcal{C}.$$

Here the first map is the graded dual of the quasi-isomorphism $\mathcal{C}^! \to \mathbf{D}(\mathcal{C})$ and is thus a quasi-isomorphism itself.

The second map is a quasi-isomorphism by theorem 4. So the composite is also, which is exactly the statement that $\mathcal{C}^!$ be Koszul. $\square$

7 Koszul complexes: the work of Priddy

We have reached the stage where we can discuss what may be regarded as the technical justification for the theory so far: elegant methods for computation of invariants from homological algebra.

For our purposes, we are interested in the Tor and Ext groups. We briefly review here their construction for propaganda purposes. However, I refer the reader to [6] (or any other book on homological algebra) for more detail, and proofs of consistency.

If A and B are a left and a right R-module respectively, for R a k-algebra, then if we take a projective resolution of A, regarded as a homologically graded module M, concentrated in nonnegative degrees, then we define

$$\mathrm{Tor}_n^R(A, B) = H_n(M \otimes_R B),$$

where B is regarded as a graded module concentrated in degree 0.

It is a standard fact that $\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^{R^{\mathrm{op}}}(B, A)$ and therefore it does not matter which of A and B we can projectively resolve, as long as we manage one or the other.

On the other hand, if we form a projective resolution M of A and an injective resolution N of B, the latter regarded this time as a cohomologically graded modules concentrated in nonnegative degrees, with $H_0 M \cong A$ and $H^0 N \cong B$, then we define:

$$\mathrm{Ext}_K^n(A, B) = H^n(\mathrm{Hom}_R(M, B)) = H^n(\mathrm{Hom}_R(A, N))$$

(where it is again a standard fact that the two right-hand objects agree).

From this we see that being able to compute small and tidy projective or injective resolutions is a very good thing indeed. Our machinery of Koszul duality affords us some good tools for doing so.

We work with graded rings and graded modules instead, where the definitions go through in the same way, but we use resolutions by projective or injective graded modules. We also make the simplification that $K = k$; with some care, this can be removed.

This section explains the classical Koszul complex for associative algebras, due to Priddy in [5]; the next section deals with the Ginzburg-Kapranov analogue for operads.

Constructing the Koszul complex

So, let $A = A_{\mathcal{A}s}(V, S)$ be an augmented quadratic associative algebra over the trivial k-algebra $A_0 = K = k$. Let R and L be a right and a left A-module respectively.

We define the **Koszul complex** $K_*(A)$ to be $A \otimes_k (A^!)^\vee \otimes_k A$, (where A is concentrated in degree 0, but $A^!$ is graded with V in degree 1). For this to be a complex of A-bimodules, we must define the differential.

Let $\Delta : (A^!)^\vee \to (A^!)^\vee \otimes (A^!)^\vee$ be the dual of the multiplication map $A^! \otimes A^! \to A^!$. Let $\lambda : (A^!)^\vee \to A$ be the map which realises the natural isomorphism $(A^!)^\vee_1 \cong V \subset A$ in degree 1, and which is zero in all other degrees.

Let also μ denote the multiplication map $A \otimes A \to A$, and s the graded sign twist map $s : (A^!)^\vee \to (A^!)^\vee$ defined by $a \mapsto (-1)^{\deg a} a$.

Now we define $d : K_*(A) \to K_*(A)$ to be the sum of the two composite maps:

$$\begin{aligned}
(1 \otimes \Delta \otimes 1) : A \otimes (A^!)^\vee \otimes A &\longrightarrow A \otimes (A^!)^\vee \otimes (A^!)^\vee \otimes A \\
(1 \otimes \lambda \otimes 1 \otimes 1) : A \otimes (A^!)^\vee \otimes (A^!)^\vee \otimes A &\longrightarrow A \otimes A \otimes (A^!)^\vee \otimes A \\
(\mu \otimes 1) : A \otimes A \otimes (A^!)^\vee \otimes A &\longrightarrow A \otimes (A^!)^\vee \otimes A,
\end{aligned}$$

and

$$\begin{aligned}
(1 \otimes s \otimes 1) : A \otimes (A^!)^\vee \otimes A &\longrightarrow A \otimes (A^!)^\vee \otimes A \\
(1 \otimes \Delta \otimes 1) : A \otimes (A^!)^\vee \otimes A &\longrightarrow A \otimes (A^!)^\vee \otimes (A^!)^\vee \otimes A \\
(1 \otimes 1 \otimes \lambda \otimes 1) : A \otimes (A^!)^\vee \otimes (A^!)^\vee \otimes A &\longrightarrow A \otimes (A^!)^\vee \otimes A \otimes A \\
(1 \otimes \mu) : A \otimes (A^!)^\vee \otimes A \otimes A &\longrightarrow A \otimes (A^!)^\vee \otimes A.
\end{aligned}$$

More comprehensibly, as inspired by [2], we can represent it as in figure 5.

The following theorem begs to be proved:

Theorem 7 *This map d is indeed a differential of A-bimodules.*

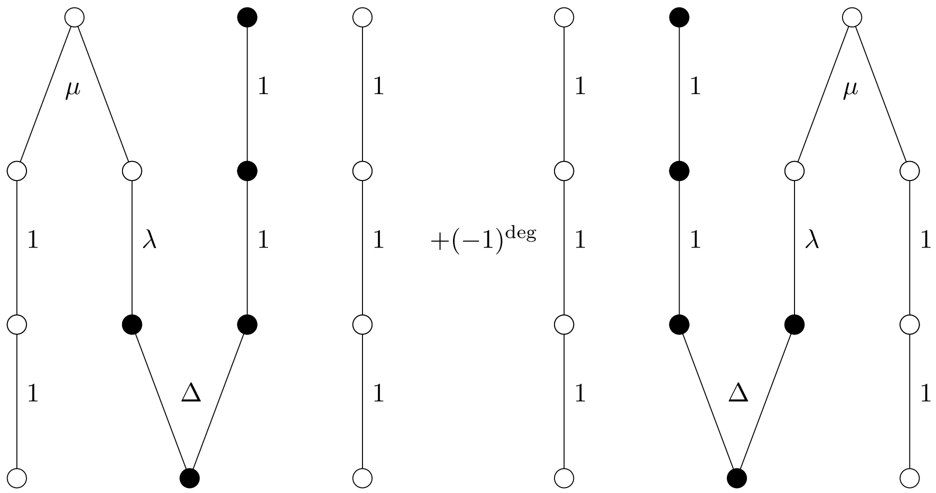

Figure 5: The differential d of the Koszul complex. Horizontal layers of dots represent tensor products of modules. White dots represent copies of A; black dots represent copies of $(A^!)^\vee$. Maps are read from bottom to top.

Proof: It is easy to see that d decreases degree by 1, and that it is a map of A-bimodules. We thus must only check that $d^2 = 0$.

The composite d^2 has four terms. I claim that product of the two left-hand terms vanishes. As figure 6 shows, it is isomorphic to something containing the map $\mu \circ (\lambda \otimes \lambda) \circ \Delta$ (portrayed as a hexagon in the right-hand figure).

But this map is zero: regarding both A and $A^!$ as graded algebras, the map is graded, and can only be non-zero in degree 2 (since λ is non-zero only in degree 1). But in degree 2 it is

$$((V \otimes V)/R)^\vee \cong (A^!)_2^\vee \to (A^!)_1^\vee \otimes (A^!)_1^\vee \cong V \otimes V \to V,$$

and this composite is zero.

As a result, the whole term vanishes.

The product of the two right-hand terms vanishes in an entirely analogous way.

The sum of the cross terms vanishes, because they both reduce to the same diagram as shown in figure 7, but with opposite signs owing to the differing placing of the map s.

So the product is zero. □

By way of example, this recipe, when applied to $A = k[x]$ yields a complex

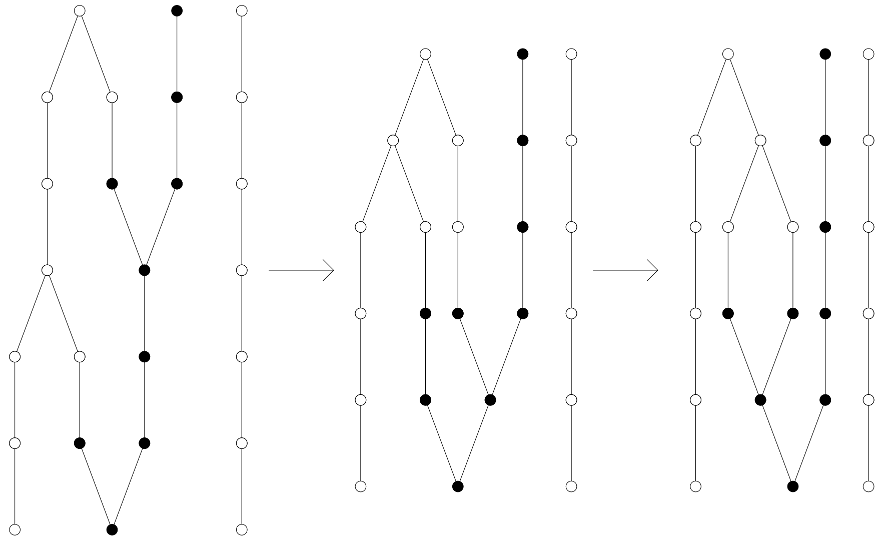

Figure 6: One term of d^2 in the Koszul complex.

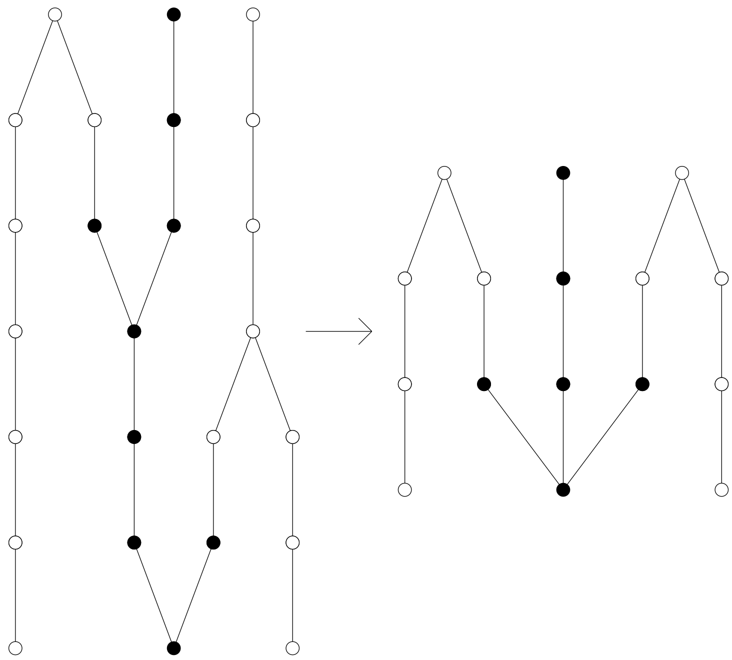

Figure 7: One cross-term in d^2 of the Koszul complex. The other one is the mirror image, and thus reduces to the same (up to sign). The map $(A^!)^\vee \to (A^!)^\vee \otimes (A^!)^\vee \otimes (A^!)^\vee$ is the dual of the multiplication map $A^! \otimes A^! \otimes A^! \to A^!$.

with one copy of $k[x] \otimes_k k[x] \cong k[x_1, x_2]$ for every basis element of $A^! = k[\theta]$ (the external algebra on θ). These form a complex:

$$0 \longleftarrow k[x_1, x_2] \xleftarrow{\times(x_1 - x_2)} k[x_1, x_2] \longleftarrow 0.$$

Conversely, when applied to $A = k[\theta]$ we get

$$0 \longleftarrow k[\theta_1, \theta_2] \xleftarrow{\times(\theta_1+\theta_2)} k[\theta_1, \theta_2] \xleftarrow{\times(\theta_1-\theta_2)} k[\theta_1, \theta_2] \xleftarrow{\times(\theta_1+\theta_2)} \cdots.$$

Note that these are both exact, save at the left where the homology is isomorphic to $k[x]$ and $k[\theta]$ respectively.

Now the general **Koszul complex** $K_*(R, A, L)$ is defined to be $R \otimes_A K_*(A) \otimes_A L$. It has the differential d induced from $K_*(A)$.

A constantly useful example is $K_*(A, A, K)$, where K is regarded as an A-module via the augmentation map.

The Koszul complex $K_*(A)$ is to be regarded as analogous to the graded dual of an operad. In a fairly analogous way to our definition of Koszul operads, we say that an algebra A is a **Koszul algebra** if $K_*(A)$ is a resolution of A. This is equivalent to many other definitions in the literature.

A quick glance at our two examples above show that $k[x]$ and $k[\theta]$ are Koszul, and it is readily proved in a similar manner that the free commutative and exterior algebras $k[x_1, \ldots, x_n]$ and $k[\theta_1, \ldots, \theta_n]$ on n generators are Koszul.

Using the Koszul complex

We have defined the Koszul property for algebras at some length. Now at last, we prove a worthwhile consequence.

The following is an important theorem of [5], establishing the computational value of the theory. I find the original proof distressing, but, as we are working in less generality, there is an easier way for us.

Theorem 8 *Let A be a Koszul algebra, and R and L modules over it. Then $H_*K_*(R, A, L) = \mathrm{Tor}_*^A(R, L)$.*

However, before we can prove this we need to introduce a piece of standard machinery: the **bar construction**. This is defined to be

$$B(A) = A \otimes_K T(I(A)) \otimes_K A,$$

where $I(A) = \bigoplus_{n \geq 1} A_n$ is the augmentation ideal of A, and $T(I(A))$ is the free associative algebra thereupon. We grade it as follows: $B_n(A) = A \otimes_K I(A)^{\otimes n} \otimes_K A$.

It is traditional to use the notation $a[a_1|\ldots|a_n]a'$ write an element of $B_n(A)$. Armed with this notation, we can also equip it with a differential:

$$\begin{aligned} d(a[a_1|\ldots|a_n]a') &= (-1)^{e_0}(aa_1)[a_2|\ldots|a_n]a' \\ &+ \sum_{i=1}^{n-1}(-1)^{e_i} a[a_1|\ldots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\ldots|a_n]a' \\ &- (-1)^{e_{n-1}} a[a_1|\ldots|a_{n-1}](a_n a'), \end{aligned}$$

where $e_i = \deg a + \deg a_1 + \cdots + \deg a_i$. Thus we are well justified in using the notation $B_*(A)$.

Then if R and L are respectively a right and a left A-module we define the **bar construction**

$$B_*(R, A, L) = R \otimes_A B(A) \otimes_A L.$$

While the bar construction and its applications could easily form the subject of another essay of this length, we will keep things self-contained, and prove everything we need as we go.

Proof (of Theorem 8):

The bar construction $B_*(A)$ is not just a DG K-module; it's a DG A-bimodule (with action on on the left by multiplying the left-hand copy of A, and action on the right by multiplying the right-hand copy of A).

Firstly, I wish to show that $B_*(A)$ is a resolution A (in the category of A-bimodules). As k is a field, $B_*(A)$ is by construction a free A-bimodule in each degree. So all that needs to be done is to show that $H_*(B_*(A)) \cong A$ in degree 0, and is A elsewhere.

I aim to do this by exhibiting a chain homotopy equivalence (see, for example, [6]) between $B_*(A)$ and A (viewed as a DG A-bimodule concentrated in degree 0).

This chain homotopy equivalence will only work in the category of right A-modules, which is good enough. So here are the required maps between

$B_*(A)$ and A:

$$\begin{array}{rcl} f : B_*(A) & \longrightarrow & A \\ a[]a' & \longmapsto & aa' \\ a[a_1|\dots|a_n]a' & \longmapsto & 0, \end{array}$$

$$\begin{array}{rcl} g : A & \longrightarrow & B_*(A) \\ a & \longmapsto & 1[]a. \end{array}$$

Now, $f \circ g = \mathrm{Id}$, and so is manifestly chain homotopic to it. It remains to show that $g \circ f \sim \mathrm{Id}$. This we do by introducing the chain homotopy

$$\begin{array}{rcl} s : B_n(A) & \longrightarrow & B_{n+1}(A) \\ a[a_1|\dots|a_n]a' & \longmapsto & (-1)^n[a|a_1|\dots|a_n]a'. \end{array}$$

It is now an elementary calculation to verify that $\mathrm{Id} - g \circ f = d \circ s + s \circ d$, completing the proof of my claim that $B_*(A)$ resolves A.

As a bonus at this point, it is clear that $B_*(A, A, L) = B_*(A) \otimes_A L$ is a free left A-module resolution of L, and so, since $B_*(R, A, L) = R \otimes_A B_*(A, A, L)$, we get that $H_n B_*(R, A, L) = \mathrm{Tor}_n^A(R, L)$.

Now, the Koszul complex $K_*(A) = A \otimes_K (A^!)^\vee \otimes_K A$ is also a DG A-bimodule that is free in each degree.

There is a canonical map $K_*(A) \to B_*(A)$. This is induced by the composite map

$$(A^!)^\vee \longrightarrow TV \longrightarrow A,$$

of the natural inclusion from $(A^!)^\vee$ into TV with the natural quotient map from TV to A.

This induces a map $K_*(R, A, L) \to B_*(R, A, L)$. So if it can be shown that the composite map $K_*(A) \to B_*(A) \to A$ is a quasi-isomorphism for A Koszul, then we are done, by same logic as in the "bonus" remark above.

But the composite is the natural map sending $K_0(A) = A \otimes_K (A^!)_0^\vee \otimes_K A \cong A \otimes_K A$ to A via $a \otimes a' \mapsto aa'$, and all other graded parts to 0. It is thus a quasi-isomorphism by the definition of a Koszul algebra. $\square$

8 Homology of algebras and Koszul operads

This section is logically necessary, not least because we have not demonstrated any examples of Koszul operads yet. However, we fulfil a grander purpose: we indicate that every quadratic operad $\mathcal{C}$ yields a naturally defined homology theory on the category of $\mathcal{C}$-algebras, then show that $\mathcal{C}$ is Koszul if and only if this theory is well-behaved in a certain sense.

Having sketched the basic theory, we show $\mathcal{A}s$, $\mathcal{C}om$ and $\mathcal{L}ie$ to be Koszul, and study the corresponding homology theories.

Homology of algebras over a quadratic operad

Let $\mathcal{C} = \mathcal{Q}(K, E, R)$ be a quadratic operad, and A any algebra over it.

Set

$$C_n(A) = (\mathcal{C}^!(n)^\vee \otimes_{K^{\otimes n}} A^{\otimes n})_{S_n}.$$

(That is, the space of coinvariants under the S_n-action).

There is a differential $d_n : C_n(A) \to C_{n-1}(A)$ on the larger collection $\mathcal{C}^!(n)^\vee \otimes_{K^{\otimes n}} A^{\otimes n}$.

The first non-trivial one, d_2, is defined as follows:

$$d_2 : f \otimes (a_1 \otimes a_2) \longmapsto 1 \otimes f(a_1, a_2).$$

Here f is an element of $\mathcal{C}^!(2)^\vee \cong E^{\vee\vee} \cong E \cong \mathcal{C}(2)$, which can thus be interpreted as a function $A^{\otimes 2} \to A$.

Higher d_n make the following diagram commute:

$$\begin{array}{ccc}
\mathcal{C}^!(n)^\vee \otimes A^{\otimes n} & \xrightarrow{\gamma^\vee} & \mathcal{C}^!(2)^\vee \otimes (\mathcal{C}^!(a)^\vee \otimes A^{\otimes a}) \otimes (\mathcal{C}^!(b)^\vee \otimes A^{\otimes b}) \\
{\scriptstyle d_n}\downarrow & & \downarrow{\scriptstyle (\pm d_a, d_b)} \\
\mathcal{C}^!(n-1)^\vee \otimes A^{\otimes n-1} & \xrightarrow[(\gamma^\vee, \gamma^\vee)]{} & \begin{array}{c} (\mathcal{C}^!(2)^\vee \otimes \mathcal{C}^!(a-1)^\vee \otimes \mathcal{C}^!(b)^\vee \otimes A^{\otimes n-1}) \\ \oplus (\mathcal{C}^!(2)^\vee \otimes \mathcal{C}^!(a)^\vee \otimes \mathcal{C}^!(b-1)^\vee \otimes A^{\otimes n-1}) \end{array}
\end{array},$$

where the sign of d_a is chosen to be $(-1)^b$.

The sign choices in this commutative diagram can be more naturally expressed using determinant vector spaces, but it is large enough already.

The diagram provides a means of computation of d, and in particular is a property that specifies d uniquely. However, it does not prove that d exists: this is done at the end of [1]. Working in the larger space $\mathcal{Q}(K, E)(n)$, it can be reduced to an unenlightening check that d respects the space $R^\perp$ of elementary ternary relations of $\mathcal{C}^!$.

To prove $d^2 = 0$, it suffices to observe that the diagram above implies this diagram:

$$\begin{array}{ccc}
\mathcal{C}^!(n)^\vee \otimes A^{\otimes n} & \xrightarrow{\gamma^\vee} & \mathcal{C}^!(2)^\vee \otimes (\mathcal{C}^!(a)^\vee \otimes A^{\otimes a}) \otimes (\mathcal{C}^!(b)^\vee \otimes A^{\otimes b}) \\
{\scriptstyle d_n^2}\downarrow & & \downarrow{\scriptstyle (d_a^2, d_b^2)} \\
\mathcal{C}^!(n-1)^\vee \otimes A^{\otimes n-1} & \xrightarrow[(\gamma^\vee,\gamma^\vee)]{} & \begin{array}{c} (\mathcal{C}^!(2)^\vee \otimes \mathcal{C}^!(a-2)^\vee \otimes \mathcal{C}^!(b)^\vee \otimes A^{\otimes n-2}) \\ \oplus(\mathcal{C}^!(2)^\vee \otimes \mathcal{C}^!(a)^\vee \otimes \mathcal{C}^!(b-2)^\vee \otimes A^{\otimes n-2}) \end{array}
\end{array}.$$

This is because the cross-terms in the differential on the right vanish as they are equal and opposite in sign. This reduces it to an easy induction.

Given that my description of d is compatible with the equivariant nature of the $C_n(A)$, it is clear that it d descends to the coinvariants to provide a differential.

Thus we may define the **homology** $H_n(A)$ of a $\mathcal{C}$-algebra by

$$H_n(A) = H_n C_*(A, d).$$

The following is proved in [1], although we shall not do so largely owing to the substantial length of the proof (in each direction).

Theorem 9 *Let $\mathcal{C} = \mathcal{Q}(K, E, R)$ be a quadratic operad. Then $\mathcal{C}$ is Koszul if and only if $H_n(A(V)) = 0$ for all $n > 1$ and all free K-modules V.*

Here $A(V)$ is the free quadratic $\mathcal{C}$-algebra generated by V (cf. section 4).

Assuming this theorem, we may go on to study our favourite operads.

Homology of associative algebras

In the notation of the subsection above, let us study $H_*(A)$ where A is an $\mathcal{A}s$-algebra.

Here the terms of the chain complex are given by

$$C_n(A) = (A^{\otimes n} \otimes_{k^{\otimes n}} \mathcal{A}s^!(n)^\vee)_{S_n} \cong (A^{\otimes n} \otimes kS_n)_{S_n} \cong A^{\otimes n},$$

where we are, of course, using that $\mathcal{A}s^! \cong \mathcal{A}s$.

The differential d_2 is given by $a_1 \otimes a_2 \mapsto a_1 a_2$, whereupon a straightforward induction proof yields

$$d_n : a_1 \otimes \cdots \otimes a_n \longmapsto \sum_i (-1)^i a_1 \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots a_n,$$

so the complex $C_*(A)$ is exactly the Hochschild complex with coefficients in k of the augmented algebra A: we have recovered Hochschild homology.

It is not hard to prove that the Hochschild homology of a free associative algebra $k\langle a_1, \ldots, a_n\rangle$ is trivial in degrees two and above. Indeed, one way to see this is as follows.

The Hochschild complex of A isomorphic to the bar complex $B_*(k, A, k)$. Thus $H_*(A) = \mathrm{Tor}^n_A(k, k)$. As $A = k\langle a_1, \ldots, a_n\rangle$, we may compute this with the Koszul complex, which gives us k in degree 0, the n-dimensional vector space $\langle a_1, \ldots, a_n\rangle$ in degree 1, and 0 in higher degrees.

Thus we conclude, by theorem 9, that $\mathcal{A}s$ is a Koszul operad.

Homology of Lie Algebras

For algebras over $\mathcal{L}ie$, the complex takes the form

$$\begin{aligned} C_n(A) &= (A^{\otimes n} \otimes_{k^{\otimes n}} \mathcal{L}ie^!(n)^\vee)_{S_n} \\ &\cong (A^{\otimes n} \otimes \mathcal{C}om(n)^\vee)_{S_n} \\ &= (A^{\otimes n} \otimes S)_{S_n} \\ &= \Lambda^n A. \end{aligned}$$

where S is, as ever, the sign representation. The differential can be easily shown to be the Eilenberg-Chevalley differential

$$d(x_1 \wedge \cdots \wedge x_n) = \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n,$$

with the hat denoting omission.

It is a standard fact that this computes the Lie algebra homology of A with constant coefficients. It is also standard that the higher Lie algebra homology of a free Lie algebra $L(V)$ is trivial, in the sense that

$$H_n(L(V)) = \begin{cases} k & \text{if } n = 0, \\ V & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Thus $\mathcal{L}ie$ is Koszul. By Theorem 6, so is $\mathcal{C}om$.

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